The theory of general relativity


Sample Solutions Exercise 9

Exercise 9.1: Electrodynamics with differential forms (6P)

In the Minkowski space with Minkowski coordinates and the metric \( g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) the electromagnetic field is encoded in a 1-form \( A \), which is related to the well-known scalar potential \( \phi \) and the vector potential \( \vec{A} = (A_x, A_y, A_z) \) by

\[
A^\sharp = \phi \partial_t + A_x \partial_x + A_y \partial_y + A_z \partial_z.
\]

(a) Write down the representation of \( A \). (1P)

(b) The electromagnetic field is defined as the 2-form \( F = dA \). Compute the components of \( F \) and express them in the components of the usual fields \( \vec{E} = -\nabla \phi - \partial_t \vec{A} \) and \( \vec{B} = \nabla \times \vec{A} \). (2P)

(c) Find and prove a simple expression for the two homogeneous Maxwell equations as a single equation using differential forms. (The result shows that the homogeneous Maxwell equations have no physical content, they rather reflect the geometric structure of the underlying exterior algebra.) (2P)

(d) Show that the electromagnetic field is invariant under a gauge transformation of the potential \( A \rightarrow A' = A + df \) for any scalar function \( f \). (1P)

Sample Solution

(a) We simply have to lower the index of \( (A^\sharp)^\mu = A^\mu \partial_\mu \): (1P)

\[
A = (A^\sharp)^b \Rightarrow A = A_\mu dx^\mu = -\phi dt + A_x dx + A_y dy + A_z dz.
\]

(b) The differential of the 1-form \( A \) is given by (1P)

\[
dA = (\partial_\mu A_\nu) \, dx^\mu \wedge dx^\nu.
\]

Writing this out we get:

\[
F = (\partial_t A_x + \partial_x \phi) \, dt \wedge dx + (\partial_t A_y + \partial_y \phi) \, dt \wedge dy + (\partial_t A_z + \partial_z \phi) \, dt \wedge dz
\]

\[
= -E_x \quad \quad -E_y \quad \quad -E_z
\]

\[
+ (\partial_x A_y - \partial_y A_x) \, dx \wedge dy + (\partial_x A_z - \partial_z A_x) \, dx \wedge dz + (\partial_y A_z - \partial_z A_y) \, dy \wedge dz
\]

\[
= B_x \quad \quad = B_y \quad \quad = B_z
\]

\[
= E_x \, dx \wedge dt + E_y \, dy \wedge dt + E_z \, dz \wedge dt
\]

\[
+ B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy.
\]

(c) The homogeneous Maxwell equations read

\[
\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} + \partial_t \vec{B} = 0.
\]

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As one can see, they are first derivatives of the fields $\vec{E}$ and $\vec{B}$. Therefore, if we want to express this relationship in terms of differential forms, we have to apply either the differential or the co-differential operator to $\mathbf{F}$. For the differential operator we get immediately $d^2 \mathbf{A} = 0$, i.e.

$$d\mathbf{F} = 0$$

This is the shortest way to write the homogeneous Maxwell equations. We compute this derivative explicitly:

$$d\mathbf{F} = \frac{1}{2} \left( \partial_{\mu} F_{\nu} \right) dx^\mu \wedge dx^\nu \wedge dx^\nu$$

$$= \frac{1}{2} \left[ (-\partial_y E_x) \, dt \wedge dx \wedge dy + (-\partial_z E_x) \, dt \wedge dx \wedge dz + \right.$$  

$$\left. (+\partial_x E_y) \, dt \wedge dx \wedge dy + (+\partial_y E_y) \, dt \wedge dy \wedge dz + \right.$$  

$$\left. (+\partial_z E_y) \, dt \wedge dx \wedge dy + (+\partial_y E_z) \, dt \wedge dy \wedge dz + \right.$$  

$$\left. (+\partial_x E_z) \, dt \wedge dx \wedge dz + (+\partial_z E_x) \, dt \wedge dy \wedge dz \right] = 0.$$  

This requires all coefficients in the four brackets to vanish. The first three brackets are just the same as the vector equation $\nabla \times \vec{E} + \partial_t \vec{B} = 0$ while the fourth bracket reproduces $\nabla \cdot \vec{B} = 0$.

(d) This is trivial in the language of differential forms:

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + df \quad \Rightarrow \quad \mathbf{F} = d\mathbf{A} \to d\mathbf{A}' = d\mathbf{A} + d^2 f = d\mathbf{A} = \mathbf{F}.$$  

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**Exercise 9.2: Holonomy**

Let $c : \lambda \in [0, \Lambda] \to \mathcal{M}$ be a parameterized curve and let $\{x^\mu\} : \mathcal{M} \to \mathbb{R}^n$ be a coordinate system on the manifold $\mathcal{M}$. Consider a tangent vector $\mathbf{Y}(0)$ at the starting point which is parallel-trans-ported along the curve.

Parallel transport means that the solution $\mathbf{Y}(\lambda) \in T_{c(\lambda)}\mathcal{M}$ is determined by the differential equation $\nabla_{\dot{c}} \mathbf{Y} = 0$, which in coordinate representation can be written as $\dot{x}^\mu \dot{Y}_\alpha + \ddot{x}^\mu \Gamma^\alpha_{\mu\nu} Y^\nu = 0$, where $\dot{x}^\mu = \frac{dx^\mu}{dt}$ are the components of the tangent vector along the curve.

(a) Show that the differential equation given above can also be written as

$$\dot{Y}_\alpha + \dot{x}^\mu \Gamma^\alpha_{\mu\nu} Y^\nu = 0.$$  

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(b) The vector $Y(\lambda)$ at the point $c(\lambda)$ is related to the initial vector $Y(0)$ by some linear map $P(\lambda) : T_0\mathcal{M} \rightarrow T_{c(\lambda)}\mathcal{M}$ with $P(0) = I$, called parallel propagator, which can be represented in coordinates as

$$Y^\alpha(\lambda) = P^\alpha_\beta(\lambda)Y^\beta(0).$$

Show that

$$P^\alpha_\beta(\lambda) = \delta^\alpha_\beta + \int_0^\lambda A^\alpha_\nu(\lambda_1)P^\nu_\beta(\lambda_1)\ d\lambda_1 \quad (*)$$

where $A^\alpha_\nu = -\Gamma^\alpha_{\mu\nu}\dot{x}^\mu$.

(c) Iterate the integral equation two times by inserting (*) into itself on the right hand side.

(d) Let us introduce the path-ordering operator $T$ which sorts the product of matrices $A(\lambda_1)A(\lambda_2)\cdots A(\lambda_n)$ with the arguments $\lambda_i$ in decreasing order, e.g.

$$T[A^\mu_\nu(\lambda_1)A^\nu_\rho(\lambda_2)] = \begin{cases} A^\mu_\nu(\lambda_1)A^\nu_\rho(\lambda_2) & \text{if } \lambda_1 \geq \lambda_2 \\ A^\mu_\nu(\lambda_2)A^\nu_\rho(\lambda_1) & \text{otherwise.} \end{cases}$$

Use this operator to rewrite the the result of (c) in such a way that all the three integrals run over the full range from 0 to $\lambda$.

(e) Generalize this result to infinitely many iterations and show that

$$P^\alpha_\beta(\lambda) = T\left[\exp\left(-\int_0^\lambda \Gamma^\alpha_{\mu\nu}(\lambda')\dot{x}^\mu(\lambda')\ d\lambda'\right)\right]$$

**Note:** If the curve is closed, this path-ordered loop integral renders a Lorentz transformation in the tangent space along the loop. This transformation is known as the holonomy of the loop. In QFT, the local parallel propagator is the so-called Wilson loop. Holonomies play an essential role in String Theory.

**Sample Solution**

(a) Writing out the covariant derivative we get

$$\dot{x}^\mu Y^\alpha_\mu = \frac{dx^\mu}{d\lambda} \partial_\mu Y^\alpha + \frac{dx^\mu}{d\lambda} \Gamma^\alpha_{\mu\nu}Y^\nu = 0.$$  

Inserting the chain rule $\dot{Y}^\alpha = \frac{\partial Y^\alpha}{\partial x^\mu} \frac{dx^\mu}{d\lambda}$ we arrive at the wanted expression.

(b) We start with the definition and take the derivative with respect to $\lambda$:

$$Y^\alpha(\lambda) = P^\alpha_\beta(\lambda)Y^\beta(0) \quad \Rightarrow \quad \dot{Y}^\alpha(\lambda) = \dot{P}^\alpha_\beta(\lambda)Y^\beta(0)$$

On the other hand, part (a) tells us that

$$\dot{Y}^\alpha(\lambda) = -\dot{x}^\mu \Gamma^\alpha_{\mu\nu}(\lambda)Y^\nu(\lambda) = -\dot{x}^\mu \Gamma^\alpha_{\mu\nu}(\lambda)P^\nu_\beta(\lambda)Y^\beta(0)$$

Comparing both expressions we are led to the ordinary differential equation

$$\dot{P}^\alpha_\beta(\lambda) = -\dot{x}^\mu \Gamma^\alpha_{\mu\nu}(\lambda)P^\nu_\beta(\lambda)$$
With the initial condition $P^\alpha_\beta(0) = \delta^\alpha_\beta$ this can be integrated on both sides, leading to the integral equation (1P)

$$P^\alpha_\beta(\lambda) = \int_0^\lambda \left( -\dot{x}^\mu(\lambda_1)\Gamma^\alpha_\mu_\nu(\lambda_1) \right) P^\nu_\beta(\lambda_1) \ d\lambda_1$$

(c) Omitting the indices (which are basically acting like matrix multiplication) the integral equation (*) can be written as

$$P(\lambda) = 1 + \int_0^\lambda d\lambda_1 A(\lambda_1) P(\lambda_1).$$

Iterating this equation two times by inserting (*) into itself we get

$$P(\lambda) = 1 + \int_0^\lambda d\lambda_1 A(\lambda_1) + \int_0^\lambda d\lambda_1 A(\lambda_1) \int_0^\lambda d\lambda_2 A(\lambda_2) P(\lambda_2).$$

(d) For commutative integrands we normally have

$$\int_0^\lambda d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \ldots = \frac{1}{2} \int_0^\lambda d\lambda_1 \int_0^\lambda d\lambda_2 \ldots.$$

Here, however, the integrand is generally non-commutative which requires to define the path-ordering operator:

$$\int_0^\lambda d\lambda_1 A(\lambda_1) \int_0^{\lambda_1} d\lambda_2 A(\lambda_2) P(\lambda_2) = \frac{1}{2} \int_0^\lambda d\lambda_1 \int_0^\lambda d\lambda_2 T\left[ A(\lambda_1) A(\lambda_2) \right] P(\lambda^*)$$

where $\lambda^* = \min(\lambda_1, \lambda_2)$.

(e) Iterating infinitely many times we end up with the series

$$P(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^\lambda d\lambda_1 \ldots \int_0^\lambda d\lambda_n T\left[ A(\lambda_1) \ldots A(\lambda_n) \right]$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} T\left[ \left( \int_0^\lambda d\lambda A(\lambda) \right)^n \right] = T\left[ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \int_0^\lambda d\lambda A(\lambda) \right)^n \right]$$

$$= T\left[ \exp\left( \int_0^\lambda d\lambda' A(\lambda') \right) \right]$$

Restoring the indices and inserting $A^\alpha_\nu = -\Gamma^\alpha_\mu_\nu \dot{x}^\mu$ we arrive at the desired result.

$$\Sigma = 12\text{P}$$