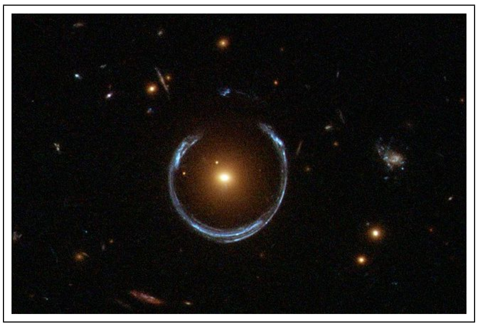
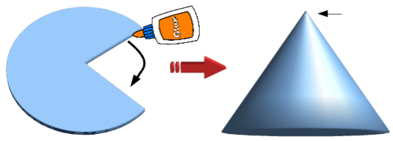
THEORY OF GENERAL RELATIVITY

LECTURE AND TUTORIALS – PROF. DR. HAYE HINRICHSEN / M.SC. ALEXANDRE ALVAREZ – WS 2019/20



Gravitational lens [Wikimeida]

EXERCISE 13.1: CONICAL SINGULARITIES / GAUSS-BONNET THEOREM[1] (5P)



Conical singularity at the tip of a cone

Mathematically, the singularity in the center of a Schwarzschild black hole is a *conical* singularity. In this exercise let us study conical singularities in the example of a simple *cone*. It is defined in polar coordinates by identification of the lines $\phi = 0$ and $\phi \to 2\pi - \delta$, where δ is called *defect angle*. The metric is chosen as the standard polar line element

$$ds^2 = dr^2 + r^2 d\phi^2,$$

which has a coordinate singularity at $r = 0$.

Because of this singularity, the Ricci scalar R can not be defined at this point in these coordinates. We can, however, calculate its value using the *Gauss-Bonnet theoerem*, which says that for any two-dimensional manifold M, with smooth boundary ∂M, we have

$$\int_M K\omega_M + \int_{\partial M} k\omega_{\partial M} = 2\pi\chi_M,$$

where the individual terms are as follows:

- K is the *Gauss curvature*, given by $K \equiv R/2$,
- k is the *geodesic curvature*, given by $k \equiv \|\nabla_u u\| = \sqrt{g(\nabla_u u, \nabla_u u)}$, where u is the normalized tangent vector along ∂M,
- ω_M and $\omega_{\partial M}$ are the natural volume forms on M and ∂M, respectively, and
- χ_M is the *Euler characteristic*.

For our purposes, we shall choose M as the cone tip $0 \leq r \leq r_0$.

(a) Look up the definition of the Euler characteristic and explain why $\chi_M = 1$. (1P)

(b) Determine $\omega_{\partial M}$, u, and k. (3P)

[1] Exercise suggested and designed by Pascal Fries

Please turn over ⇒

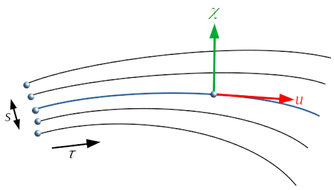
(c) Use your results to show that

$$\int_M R\omega_M = 2\delta,$$

independently of r_0. Why does this mean that $r = 0$ is a *physical* singularity (for $\delta \neq 0$) and not only a coordinate singularity? (1P)

Exercise 13.2: Geodesic deviation (7P)

Consider a bundle of nearby geodesic lines $c(s, \tau)$ labeled by a continuous parameter $s \in \mathbb{R}$. In a given gravitational field, these trajectories will diverge or converge, as sketched in the figure. In a certain point given by s and the eigenzeit τ, we define



- the 4-velocity $u(s,\tau) = \frac{\partial}{\partial \tau} c(s,\tau) \in T\mathcal{M}$
- the distance vector $\chi(s,\tau) = \frac{\partial}{\partial s} c(s,\tau) \in T\mathcal{M}$

In this setting, the *deviation velocity* is defined as the first covariant derivative $v = \nabla_u \chi$ while the *deviation acceleration* is defined as the second covariant derivative $a = \nabla_u \nabla_u \chi$.

(a) Compute the deviation velocity components v^α using the identity $\frac{\mathrm{d}}{\mathrm{d}\tau} = u^\beta \partial_\beta$. (1P)

(b) Compute the deviation acceleration components a^α and eliminate v^α and $\dot{v}^\alpha$ by using the eigenzeit derivative of (a). Note that all contributions (u, χ and the Christoffel symbols) depend on the position and therewith on τ. (2P)

(c) Use the fact that the geodesic equation holds for all geodesic lines in order to show that the geodesic deviation evolves according to the differential equation (2P)

$$\ddot{\chi}^\alpha + 2\Gamma^\alpha{}_{\beta\gamma} u^\beta \dot{\chi}^\gamma + \left(\partial_\delta \Gamma^\alpha{}_{\beta\gamma}\right) u^\beta u^\gamma \chi^\delta = 0$$

(d) Insert $\ddot{\chi}^\alpha$ from (c) into the result of (b) in order to to eliminate $\ddot{\chi}$ and use the geodesic equation to eliminate $\dot{u}$. (1P)

(e) Demonstrate that the result from (d) is the coordinate representation of the abstract equation

$$\boxed{a = R(u,\chi)u}$$

where R is the Riemann curvature tensor. (1P)

Remark: The acceleration a can be used to determine the so-called *tidal forces*.

($\Sigma = 12$P)

Last exercise sheet. To be handed in on Wednesday, January 29, at the beginning of the tutorial.