Theory of General Relativity

Lecture and Tutorials – Prof. Dr. Haye Hinrichsen / M.Sc. Alexandre Alvarez – WS 2019/20

Dust

Exercise 12.1: Energy-momentum tensor for dust (4P)

The energy-momentum tensor for dust is $T^{\mu\mu} = \rho u^\mu u^\nu$, where u^μ with $u^\mu u_\mu = -1$ is the velocity of the dust relative to the observer and ρ is the energy density seen from the observers frame.

(a) Show that the local conservation of energy-momentum $T^{\mu\nu}{}_{;\mu} = 0$ implies that the dust particles move on geodesic lines. (2P)

(b) If the dust does not rotate one can find a co-moving coordinate system with $g_{0i} = 0, i = 1, 2, 3$. If one assumes spherical symmetry, the line element has to be of the form

$$\mathrm{d}s^2 = -e^{2A}\,\mathrm{d}t^2 + e^{2B}\,\mathrm{d}r^2 + e^{2C}\,\mathrm{d}\Omega^2\,,$$

where A, B, C depend only on t and r. Show that it is always possible to transform the time coordinate in such a way that $A = 0$. What would be the interpretation of the new time coordinate? (2P)

Exercise 12.2: Energy momentum tensor in the moving frame (2P)

Assume that the energy momentum tensor of a gas in the co-moving frame is given by

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}.$$

Show that for an observer moving relative to this frame with the four-velocity u^μ the energy-momentum tensor will be given by

$$T^{\mu\nu} = (\rho + p/c^2) u^\mu u^\nu + p g^{\mu\nu}$$

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Exercise 12.3: Static Universe **(6P)**

In the case of a homogeneous Universe the Einstein equations reduce to the so-called Friedmann equations

$$(I) \qquad \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{kc^2}{a(t)^2} + \frac{\Lambda c^2}{3}$$

$$(II) \qquad \frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\left(\rho(t) + \frac{3p(t)}{c^2}\right) + \frac{\Lambda c^2}{3}$$

Here G is the gravitational constant, t is time, $\rho(t)$ and $p(t)$ are the mass density and the pressure of the matter in the Universe, $a(t) > 0$ is the scale factor proportional to the "size" of the Universe, $k = 0, \pm 1$ is the curvature parameter, Λ is the cosmological constant, and c is the velocity of light. In the following let us consider of the case of a Universe with positive curvature $k = +1$ filled by dust, that is $\rho(t) > 0$ and $p(t) = 0$. Moreover, let t_0 be the current time and $\rho(t_0)$ the current mass density of the Universe.

(a) Eliminate $\rho(t)$ to obtain a single differential equation. (2P)

(b) Show that for $\Lambda = 0$ and $\dot{a}(t_0) = 0$ the Universe would shrink monotonously. (1P)

(c) Choose the cosmological constant in such a way that it stabilizes a static Universe. Express Λ and $a_0 = a(t_0)$ in terms of $\rho_0 = \rho(t_0)$. (1P)

(d) Consider a small deviation $a(t) = a_0 + \epsilon(t)$, $\rho(t) = \rho_0 + \delta(t)$ and linearize the differential equations to lowest order in these deviations. Solve the linearized differential equations in order to analyze the stability of the static solution. (2P)

(Σ = 12P)

To be handed in on Wednesday, January 22, at the beginning of the tutorial.