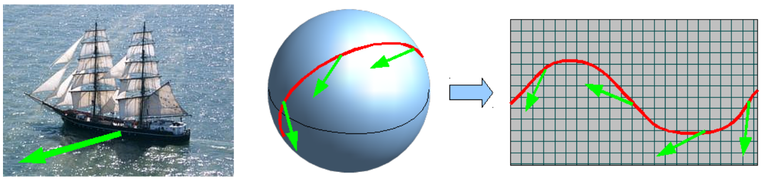

Parallel transport in reality and on a map.

Exercise 11.1: Levi-Civitá connection (3P)

The connection $\nabla_j \mathbf{Y} = \nabla_{\mathbf{e}_j} \mathbf{Y}$ is defined as the rate at which vector field $\mathbf{Y}$ changes if we move in the direction of $\mathbf{e}_j$. In an arbitrary basis this rate can be represented as

$$\nabla_{\mathbf{e}_j} \mathbf{e}_i = \Gamma^k_{ij} \mathbf{e}_k \,.$$

A connection is called *Levi-Civitá connection* if it is

- compatible with the metric ($\nabla_{\mathbf{X}} \mathbf{g} = 0$)
- torsion-free ($\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0$).

(a) What are the symmetries of the indices in the representation of a Levi-Civitá connection? How many independent components of Γ do we have in two dimensions? (1P)

(b) Compute the components of such a connection with respect to the standard coordinate basis of $\mathbb{R}^2$ with the metric

$$\mathbf{g} = \mathbf{dx} \otimes \mathbf{dx} + f(x) \mathbf{dy} \otimes \mathbf{dy}$$

using the two relations given above, but without using the explicit formula for the Christoffel symbols in terms of g that was given in the lecture. (2P)

Exercise 11.2: Levi-Civitá connection and geodesics (3P)

Solve the differential equation for the geodesics in the previous exercise for the special cases $f(x) = x^2$ and $f(x) = \sin^2 x$.

Please turn over ⇒

Exercise 11.3: Rindler observers and Rindler coordinates[1] (6P)


Wolfgang Rindler

The world line γ of a *Rindler observer* in 2-dimensional Minkowski space is given by

$$t(\tau) = \alpha^{-1} \sinh \alpha\tau, \quad x(\tau) = \alpha^{-1} \cosh \alpha\tau,$$

where α is a constant.

(a) Show that τ measures the proper time (Eigenzeit) along γ. Then, calculate the (proper) acceleration $a \equiv \nabla_{\frac{\mathrm{d}}{\mathrm{d}\tau}} \frac{\mathrm{d}}{\mathrm{d}\tau}$ and check that it satisfies $\|a\| = |\alpha|$. (1P)

(b) We now go to the rest frame of a Rindler observer with acceleration α. A convenient choice of coordinates is $\{\tau, R\}$, defined by

$$t = \alpha^{-1} R \sinh \alpha\tau, \quad x = \alpha^{-1} R \cosh \alpha\tau.$$

Draw a spacetime diagram of Minkowski space with coordinates $\{t, x\}$ and sketch the lines of constant τ, R. Furthermore, determine the form of the line element $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$ in terms of $\mathrm{d}\tau, \mathrm{d}R$. (2P)

(c) For fixed τ, calculate the proper distance between the two points (τ, R_1) and (τ, R_2). *Hint: Check the line element from above – there is an isometry which makes this calculation trivial!* (1P)

(d) Consider now a second observer with identical acceleration α, whose world line $\tilde{\gamma}$ is given by

$$t(\tau') = \alpha^{-1} \sinh \alpha\tau', \quad x(\tau') = \Delta x + \alpha^{-1} \cosh \alpha\tau'.$$

Include both γ and $\tilde{\gamma}$ in your spacetime diagram from above. Calculate the proper distance between the observers at *coordinate time* $\tau = 0$. Then calculate it for general coordinate time τ. What happens in the limit $\tau \to \infty$? (2P)

(Σ = 12P)

To be handed in on Wednesday, January 15, at the beginning of the tutorial.

[1]Exercise suggested and designed by Pascal Fries