Exercise 11.1: Levi-Civitá connection  (3P)

The connection $\nabla_j Y = \nabla_{e_j} Y$ is defined as the rate at which vector field $Y$ changes if we move in the direction of $e_j$. In an arbitrary basis this rate can be represented as

$$\nabla_{e_j} e_i = \Gamma^k_{ij} e_k .$$

A connection is called *Levi-Civitá connection* if it is

- compatible with the metric ($\nabla_X g = 0$)
- torsion-free ($\nabla_X Y - \nabla_Y X - [X,Y] = 0$).

(a) What are the symmetries of the indices in the representation of a Levi-Civitá connection? How many independent components of $\Gamma$ do we have in two dimensions?  (1P)

(b) Compute the components of such a connection with respect to the standard coordinate basis of $\mathbb{R}^2$ with the metric

$$g = dx \otimes dx + f(x)dy \otimes dy$$

using the two relations given above, but without using the explicit formula for the Christoffel symbols in terms of $g$ that was given in the lecture.  (2P)

Exercise 11.2: Levi-Civitá connection and geodesics  (3P)

Solve the differential equation for the geodesics in the previous exercise for the special cases $f(x) = x^2$ and $f(x) = \sin^2 x$. 

Please turn over ⇒
Exercise 11.3: Rindler observers and Rindler coordinates

The world line $\gamma$ of a Rindler observer in 2-dimensional Minkowski space is given by

$$t(\tau) = \alpha^{-1} \sinh \alpha \tau, \quad x(\tau) = \alpha^{-1} \cosh \alpha \tau,$$

where $\alpha$ is a constant.

(a) Show that $\tau$ measures the proper time (Eigenzeit) along $\gamma$. Then, calculate the (proper) acceleration $a \equiv \nabla \frac{d}{d\tau}$ and check that it satisfies $\|a\| = |\alpha|$. (1P)

(b) We now go to the rest frame of a Rindler observer with acceleration $\alpha$. A convenient choice of coordinates is $\{\tau, R\}$, defined by

$$t = \alpha^{-1} R \sinh \alpha \tau, \quad x = \alpha^{-1} R \cosh \alpha \tau.$$

Draw a spacetime diagram of Minkowski space with coordinates $\{t, x\}$ and sketch the lines of constant $\tau, R$. Furthermore, determine the form of the line element $ds^2 = -dt^2 + dx^2$ in terms of $d\tau, dR$. (2P)

(c) For fixed $\tau$, calculate the proper distance between the two points $(\tau, R_1)$ and $(\tau, R_2)$. Hint: Check the line element from above – there is an isometry which makes this calculation trivial! (1P)

(d) Consider now a second observer with identical acceleration $\alpha$, whose world line $\tilde{\gamma}$ is given by

$$t(\tau') = \alpha^{-1} \sinh \alpha \tau', \quad x(\tau') = \Delta x + \alpha^{-1} \cosh \alpha \tau'.$$

Include both $\gamma$ and $\tilde{\gamma}$ in your spacetime diagram from above. Calculate the proper distance between the observers at coordinate time $\tau = 0$. Then calculate it for general coordinate time $\tau$. What happens in the limit $\tau \to \infty$? (2P)

($\Sigma = 12P$)

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To be handed in on Wednesday, January 15, at the beginning of the tutorial.

1Exercise suggested and designed by Pascal Fries