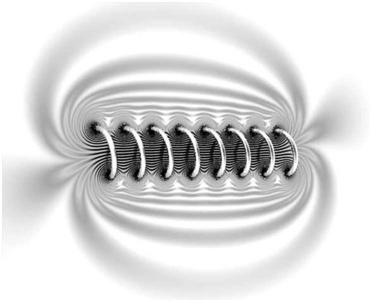

Solenoid [Wikimedia]

Exercise 9.1: Electrodynamics with differential forms (6P)

In the Minkowski space with Minkowski coordinates and the metric $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ the electromagnetic field is encoded in a 1-form $\mathbf{A}$, which is related to the well-known scalar potential ϕ and the vector potential $\vec{A} = (A_x, A_y, A_z)$ by

$$\mathbf{A}^\sharp = \phi\partial_t + A_x\partial_x + A_y\partial_y + A_z\partial_z.$$

(a) Write down the representation of $\mathbf{A}$. (1P)

(b) The electromagnetic field is defined as the 2-form $\mathbf{F} = \mathrm{d}\mathbf{A}$. Compute the components of $\mathbf{F}$ and express them in the components of the usual fields $\vec{E} = -\nabla\phi - \partial_t\vec{A}$ and $\vec{B} = \nabla \times \vec{A}$. (2P)

(c) Find and prove a simple expression for the two homogeneous Maxwell equations as a single equation using differential forms. *(The result shows that the homogeneous Maxwell equations have no physical content, they rather reflect the geometric structure of the underlying exterior algebra.)* (2P)

(d) Show that the electromagnetic field is invariant under a gauge transformation of the potential $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \mathrm{d}f$ for any scalar function f. (1P)

Exercise 9.2: Holonomy (6P)

Let $c : \lambda \in [0, \Lambda] \to \mathcal{M}$ be a parameterized curve and let $\{x^\mu\} : \mathcal{M} \to \mathbb{R}^n$ be a coordinate system on the manifold $\mathcal{M}$. Consider a tangent vector $\mathbf{Y}(0)$ at the starting point which is parallel-transported along the curve. Parallel transport means that the solution $\mathbf{Y}(\lambda) \in T_{c(\lambda)}\mathcal{M}$ is determined by the differential equation $\nabla_{\dot{x}}\mathbf{Y} = 0$, which in coordinate representation can be written as $\dot{x}^\mu Y^\alpha_{\ ;\mu} = 0$, where $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ are the components of the tangent vector along the curve.

(a) Show that the differential equation given above can also be written as (1P)

$$\dot{Y}^\alpha + \dot{x}^\mu \Gamma^\alpha_{\ \mu\nu} Y^\nu = 0.$$

(b) The vector $\mathbf{Y}(\lambda)$ at the point $c(\lambda)$ is related to the initial vector $\mathbf{Y}(0)$ by some linear map $\mathbf{P}(\lambda) : T_0\mathcal{M} \to T_{c(\lambda)}\mathcal{M}$ with $\mathbf{P}(0) = \mathbb{1}$, called *parallel propagator*, which

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can be represented in coordinates as

$$Y^\alpha(\lambda) = P^\alpha{}_\beta(\lambda) Y^\beta(0).$$

Show that

$$P^\alpha{}_\beta(\lambda) \;=\; \delta^\alpha_\beta + \int_0^\lambda A^\alpha{}_\nu(\lambda_1) P^\nu{}_\beta(\lambda_1)\,\mathrm{d}\lambda_1 \qquad (*)$$

where $A^\alpha{}_\nu = -\Gamma^\alpha{}_{\mu\nu}\dot{x}^\mu$. (2P)

(c) Iterate the integral equation two times by inserting (*) into itself on the right hand side. (1P)

(d) Let us introduce the *path-ordering operator* $\mathcal{T}$ which sorts the product of matrices $A(\lambda_1)A(\lambda_2)\cdots A(\lambda_n)$ with the arguments λ_i in decreasing order, e.g.

$$\mathcal{T}\Big[A^\mu{}_\nu(\lambda_1)A^\nu{}_\rho(\lambda_2)\Big] = \begin{cases} A^\mu{}_\nu(\lambda_1)A^\nu{}_\rho(\lambda_2) & \text{if } \lambda_1 \geq \lambda_2 \\ A^\mu{}_\nu(\lambda_2)A^\nu{}_\rho(\lambda_1) & \text{otherwise.} \end{cases}$$

Use this operator to rewrite the the result of (c) in such a way that all the three integrals run over the full range from 0 to λ. (1P)

(e) Generalize this result to infinitely many iterations and show that (1P)

$$P^\alpha{}_\beta(\lambda) \;=\; \mathcal{T}\Big[\exp\Big(-\int_0^\lambda \Gamma^\alpha{}_{\mu\nu}(\lambda')\dot{x}^\nu(\lambda')\,\mathrm{d}\lambda'\Big)\Big]$$

Note: If the curve is closed, this path-ordered loop integral renders a Lorentz transformation in the tangent space along the loop. This transformation is known as the **holonomy** of the loop. In QFT, the local parallel propagator is the so-called **Wilson loop**. Holonomies play an essential role in String Theory.

(Σ = 12P)

To be handed in on Wednesday, December 18, at the beginning of the tutorial.