In the tetrad basis [PxHere, public domain]

EXERCISE 8.1: LOGARITHMIC DERIVATIVE (2P)

Prove that $\Gamma^{\mu}{}_{\nu\mu} = \partial_\nu \ln \sqrt{|g|}$.

EXERCISE 8.2: TETRAD FORMALISM AND THE SPIN CONNECTION (6P)

Let $\{x^\mu\} = \{x^1, \dots, x^n\}$ be a coordinate system of an n-dimensional manifold $\mathcal{M}$ and denote by $\{\partial_\mu\}$ and $\{\,\mathrm{d}x^\nu\}$ the associated basis vector fields of $T\mathcal{M}$ and $T^*\mathcal{M}$, respectively. Let $\mathcal{M}$ be equipped with the position-dependent metric $\mathbf{g} = g_{\mu\nu}(x)\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$.

Then, at each point of the manifold, we can introduces a set basis vectors fields $\mathbf{e}_i(x)$, called *vielbein* or *tetrad*, in which the metric becomes orthonormal, i.e., it ĺocally equals the Euclidean or Minkowski metric $\mathbf{g}(\mathbf{e}_i, \mathbf{e}_j) = \eta_{ij}$. In GR, this basis corresponds to the local free-fall basis.

At each point on $\mathcal{M}$ we can represent the tetrad basis vector field in terms of the coordinate basis by $\mathbf{e}_i = e_i{}^\mu \partial_\mu$ and $\mathbf{e}^j = e^j{}_\nu\,\mathrm{d}x^\nu$ with certain coefficients $e_i{}^\mu(x)$ and $e^j{}_\nu(x)$. Likewise there are inverse relations $\partial_\nu = e_\nu{}^j \mathbf{e}_j$ and $\mathrm{d}x^\mu = e^\mu{}_i \mathbf{e}^i$ with coefficients $e_\nu{}^j(x)$ and $e^\mu{}_i(x)$.

(a) Show that the coefficients are symmetric under transposition, i.e., $e_i{}^\mu(x) = e^\mu{}_i(x)$ and $e^j{}_\nu(x) = e_\nu{}^j(x)$. (1P)

(b) The coefficients $e^j{}_\nu$ can be thought of as being the components of a $(1,1)$-tensor $\mathbf{E} = e^j{}_\nu\, \mathbf{e}_j \otimes \mathrm{d}x^\nu$. What does this form, interpreted as a linear map from the tangent space onto itself, do with an abstract tangent vector? (1P)

(c) Express the connection coefficients $\Gamma^a{}_{ij}$ in the tetrad basis in terms of the connection coefficients (Christoffel symbols) $\Gamma^\alpha{}_{\mu\nu}$ in the coordinate basis. (2P)
Hint: Represent $\nabla_\mathbf{X}\mathbf{Y}$ in coordinate and tetrad basis and compare the components.

(d) True relativity geeks prefer to work with the *spin connection* ω, a Lorentz-algebra-valued 1-form defined by the components $\omega_\mu{}^{ab} := e_\mu{}^j \Gamma^a{}_{ij} \eta^{bi}$. Show that (2P)

$$\omega_\mu{}^{ab} = e^a{}_\nu\, \eta^{bi}\, e^\nu{}_{i;\mu}\ .$$

Remark: This connection can be interpreted as follows: If we go a in direction $\mathbf{X}$, then $\omega(\mathbf{X})$ is the corresponding generator of the Lorentz transformation in the flat tetrad space that can be used to go from one tangent space to the next.

Please turn over ⇒

EXERCISE 8.3: GEODESIC LINE IN THE UPPER POINCARÉ HALF PLANE (4P)

Consider the upper half plane $U = \{(x, y)|\ y > 0\}$ in two dimensions with the metric

$$\mathbf{g} = y^{-2}\,(\mathbf{dx} \otimes \mathbf{dx} + \mathbf{dy} \otimes \mathbf{dy}) \ .$$

(a) Determine the geodesic equations. (1P)

(b) Solve the geodesic equations in arc length parameterization. You may use *Mathematica*® or similar tools. (2P)

(c) What is the form of the geodesic lines? What is their maximal length? (1P)

(Σ = **12P**)

To be handed in on Wednesday, December 11, at the beginning of the tutorial.