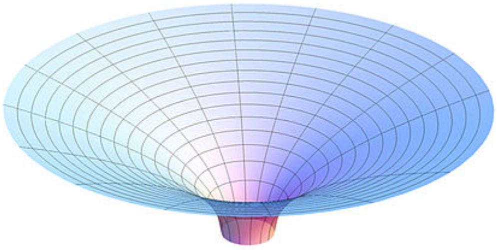

Schwarzschild metric [Wikimedia]

Exercise 10.1: Principle of Least Action (6P)

The electromagnetic current is a 3-form $\mathbf{J}$ which is related to the usual charge density ρ and the current $\vec{J} = (J_x, J_y, J_z)$ by

$$(\star\mathbf{J})^\sharp = \rho\partial_t + J_x\partial_x + J_y\partial_y + J_z\partial_z\,.$$

The action S of the electromagnetic field contained in a 4-volume G of the Minkowski space is given by $S = \int_G \mathbf{L}$, where $\mathbf{L}$ is a 4-form called *Lagrange density*. In electrodynamics, it is given by

$$\mathbf{L} := -\frac{1}{2}\,\mathrm{d}\mathbf{A}\wedge\star\mathrm{d}\mathbf{A} \,+\, \mathbf{A}\wedge\mathbf{J}.$$

Now consider an infinitesimal variation of the electromagnetic field by $\mathbf{A} \to \mathbf{A}+\epsilon\mathbf{Q}$, where $\mathbf{Q}$ is a 1-form and where $0 < \epsilon \ll 1$ is small. The principle of least action tells us that S does not change to lowest order in ϵ, provided that $\mathbf{Q}$ vanishes on the boundary ∂G.

(a) Show that the stationarity of S leads us to the condition (2P)

$$\int_G \Big(-\,\mathrm{d}\mathbf{Q}\wedge\star\mathrm{d}\mathbf{A} + \mathbf{Q}\wedge\mathbf{J}\Big) = 0. \qquad\qquad \forall\mathbf{Q}$$

(b) Use Stokes theorem to show that $\int_G(\,\mathrm{d}\mathbf{Q}\wedge\star\mathrm{d}\mathbf{A}) = \int_G(\mathbf{Q}\wedge\,\mathrm{d}\star\,\mathrm{d}\mathbf{A})$, using the fact that $\mathbf{Q}\big|_{\partial G} = 0$. Insert this relation into the result of (f) to show that (2P)

$$\int_G \mathbf{Q}\wedge\big(-\,\mathrm{d}\star\,\mathrm{d}\mathbf{A} + \mathbf{J}\big) = 0 \qquad \forall\mathbf{Q} \text{ with } \mathbf{Q}\big|_{\partial G} = 0.$$

(c) Show that the above result leads to the inhomogeneous Maxwell equations (1P)

$$\boxed{\mathrm{d}\star\mathbf{F} = \mathbf{J}.}$$

(d) Show that with the Lorenz gauge $\mathrm{d}^\dagger\mathbf{A} = 0$ the inhomogeneous Maxwell equations imply the wave equation $\Box\mathbf{A} = \star\mathbf{J}$, where $\Box = \mathrm{d}\,\mathrm{d}^\dagger + \mathrm{d}^\dagger\,\mathrm{d}$ is the Laplace-de-Rham operator in Minkowski space. (1P)

Please turn over ⇒

Exercise 10.2: Trajectories in the Schwarzschild metric (6P)

The line element of the Schwarzschild space time in vacuum ($T_{\mu\nu} = 0$) in Schwarzschild coordinates is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \,.$$

(a) The Lagrange function of a free particle with mass m is given by $L = -\frac{m}{2} u^\mu u_\mu$. Use the Euler-Lagrange equations to show that the corresponding canonical momenta $p_t = \partial L/\partial \dot{t}$ and $p_\phi = \partial L/\partial \dot{\phi}$ are conserved. (1P)

(b) Show that the initial conditions $\tau = \tau_0, \quad \theta(\tau_0) = \pi/2$, and $\dot{\theta}(\tau_0) = 0$ ensure planar motion, meaning that $\theta = const$ for all τ. (1P)

(c) If the trajectory is parametrized by the proper time (eigenzeit), then $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$. Show that this relation can be rewritten as $\frac{1}{2} m \dot{r}^2 + V(r) = E$ and determine $V(r)$ and E. To what extent does $V(r)$ differ from the corresponding potential in the Newtonian theory? Sketch both potentials qualitatively. (2P)

(d) Consider a particle falling freely in radial direction with the initial conditions $\tau = \tau_0, \dot{r}(\tau_0) = \dot{\phi}(\tau_0) = \dot{\theta}(\tau_0) = t(\tau_0) = 0$. How much eigenzeit elapses until the particle reaches the Schwarzschild radius? (2P)

(Σ = 12P)

To be handed in on Wednesday, January 08, at the beginning of the tutorial.