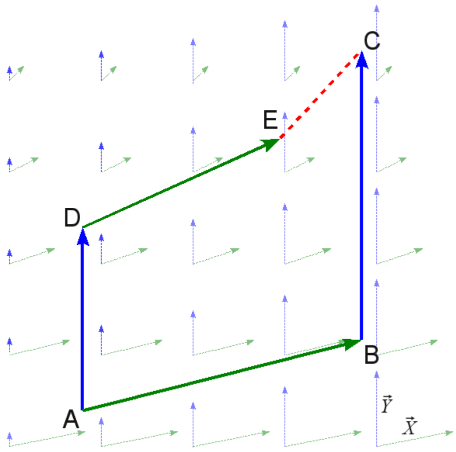

The Lie bracket detects non-commuting paths

Exercise 7.1: Exact and closed forms (2P)

Which of the following differential 1-forms is closed and which are exact? In the case of exact forms, try to guess the corresponding potential function. (each 0.5P)

(a) $\alpha = 3y\,\mathrm{d}x + x\,\mathrm{d}y$

(b) $\alpha = y\,\mathrm{d}x$

(c) $\alpha = e^x y\,\mathrm{d}x + e^x\,\mathrm{d}y$

(d) $\alpha = -y\,\mathrm{d}x + x\,\mathrm{d}y$

Exercise 7.2: Volume form in spherical coordinates (3P)

The transformation from spherical back to Cartesian coordinates reads

$$x = r\cos\phi\sin\theta\,, \qquad y = r\sin\phi\sin\theta\,, \qquad z = r\cos\theta\,.$$

(a) Express the basis $\{\mathbf{f}_m \,|\, m \in \{r,\,\theta,\,\phi\}\}$ in terms of the Cartesian basis $\{\mathbf{e}_i \,|\, i \in \{x,\,y,\,z\}\}$. Hint: Basis vectors can be understood as partial derivatives. (1P)

(b) Calculate the components of the metric. (1P)

(c) Compute the volume form in the spherical basis. (1P)

Exercise 7.3: Laplace-Beltrami operator acting on functions (3P)

Let $f : U \to \mathbb{R}$ be a scalar function on $U \in \mathbb{R}^n$ with an arbitrary metric $\mathbf{g}$. Furthermore let $x^1, \ldots x^n$ be an arbitrary coordinate system on U with the associated coordinate basis.

(a) Compute $\Delta f = (\mathbf{d} + \mathbf{d}^\dagger)^2 f$. (2P)

(b) Specialize the result from (a) to the case of sperical coordinates r, θ, ϕ in $\mathbb{R}^3$. (1P)

Please turn over ⇒

Exercise 7.4: Lie derivative **(4P)**

The Lie derivative $\mathcal{L}_{\mathbf{X}}$ with respect to a differentiable vector fiels $\mathbf{X}$ is defined as follows:

- On scalar fields f the Lie derivative acts as an ordinary directional derivative:

$$\mathcal{L}_{\mathbf{X}} f = \mathbf{X} f$$

- On vector fields $\mathbf{Y}$ it is given by the Lie bracket (see lecture notes 2.4.7 on page 63):

$$\mathcal{L}_{\mathbf{X}} \mathbf{Y} = [\mathbf{X}, \mathbf{Y}] = \mathbf{X} \circ \mathbf{Y} - \mathbf{Y} \circ \mathbf{X}$$

Assume that all vector fields and functions are sufficiently often differentiable.
Prove the follwing relations:

(a) Show that $[\mathcal{L}_{\mathbf{X}}, \mathcal{L}_{\mathbf{Y}}] = \mathcal{L}_{[\mathbf{X},\mathbf{Y}]}$ acting on functions as well as on vector fields.

(b) Prove the Jacobi identity $[[\mathcal{L}_{\mathbf{X}}, \mathcal{L}_{\mathbf{Y}}], \mathcal{L}_{\mathbf{Z}}] + [[\mathcal{L}_{\mathbf{Y}}, \mathcal{L}_{\mathbf{Z}}], \mathcal{L}_{\mathbf{X}}] + [[\mathcal{L}_{\mathbf{Z}}, \mathcal{L}_{\mathbf{X}}], \mathcal{L}_{\mathbf{Y}}] = 0.$

(c) Let $\mathbf{e}_j$ be an arbitrary basis vector field of the tangent space. Interpreting the basis fields as directional derivatives, show that

$$\mathcal{L}_{\mathbf{X}} \mathbf{Y} = X^i(\mathbf{e}_i Y^j)\mathbf{e}_j - Y^i(\mathbf{e}_i X^j)\mathbf{e}_j + X^i Y^j [\mathbf{e}_i, \mathbf{e}_j]\,.$$

(d) Show that in the special case of a coordinate basis this expression simplifies to

$$\mathcal{L}_{\mathbf{X}} \mathbf{Y} = X^\mu(\partial_\mu Y^\nu)\partial_\nu - Y^\mu(\partial_\mu X^\nu)\partial_\nu\,.$$

(Σ = 12P)

To be handed in on Wednesday, December 04, at the beginning of the tutorial.