Exercise 5.1: Wedge product and determinants (5P)

Let V be a d-dimensional vector space and let $\{\mathbf{e}_i\}$ be a basis of V. Furthermore, let $\mathbf{v}_{(1)}, \ldots, \mathbf{v}_{(n)}$ be vectors in V.

(a) Show that if two of these vectors are equal, then the wedge product $\mathbf{v}_{(1)} \wedge \ldots \wedge \mathbf{v}_{(n)}$ vanishes. (1P)

(b) Show that $\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_n} = \epsilon_{i_1,\ldots,i_n} \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n$. (1P)

(c) Prove that for $n = d$ the wedge product of the vectors is given by

$$\mathbf{v}_{(1)} \wedge \ldots \wedge \mathbf{v}_{(d)} = c\, \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_d,$$

where c is the determinant of the matrix containing the vector components. (2P)

(d) What is the dimension of the vector space spanned by all wedge products of the form $\mathbf{v}_{(1)} \wedge \ldots \wedge \mathbf{v}_{(d)}$? (1P)

Exercise 5.2: Adjoint linear map (2P)

Consider a linear map $\mathbf{A} : V \to V$ represented in a given basis by components $A^i{}_j$. The adjoint linear map $\mathbf{A}^\dagger$ is defined by

$$\mathbf{g}(\mathbf{v}, \mathbf{A}\mathbf{w}) \;=\; \mathbf{g}(\mathbf{A}^\dagger \mathbf{v}, \mathbf{w}) \qquad \forall \mathbf{v}, \mathbf{w} \in V.$$

Compute the components $A^{\dagger k}{}_l$.

Exercise 5.3: Hodge duality (5P)

Let $\{\mathbf{e}_x\,, \mathbf{e}_y\,, \mathbf{e}_z\}$ be the standard basis in the Euclidean $\mathbb{R}^3$. A given vector $\mathbf{a} = a^i \mathbf{e}_i$ can be associated with a form in two different ways:

- We can associate with this vector a 1-form $\mathbf{a}^\flat$ by applying the musical isomorphism.
- We can associate with this vector a 2-form $\star\mathbf{a}^\flat$ which is the Hodge dual of the 1-form given above.

(a) Let $\mathbf{a} = \mathbf{e}_x + 2\mathbf{e}_y - \mathbf{e}_z$ and $\mathbf{b} = 2\mathbf{e}_x - 3\mathbf{e}_y + \mathbf{e}_z$. Determine the 1-forms $\mathbf{a}^\flat$ and $\mathbf{b}^\flat$ as well as the 2-forms $\star\mathbf{a}^\flat$ and $\star\mathbf{b}^\flat$. (2P)

(b) Now let $\mathbf{a}$ and $\mathbf{b}$ be arbitrary vectors. Show that

$$\left(\mathbf{a}^\flat \wedge \mathbf{b}^\flat\right)_{ij} = \epsilon_{ijk} C^k\,,$$

where $\mathbf{C} = \mathbf{a} \times \mathbf{b}$ is the standard cross product, and show that (2P)

$$\mathbf{a}^\flat \wedge \star\mathbf{b}^\flat = \mathbf{g}\,(\mathbf{a}, \mathbf{b})\, \mathbf{e}^x \wedge \mathbf{e}^y \wedge \mathbf{e}^z\,.$$

(c) Find also the Hodge dual of the 1-form $\mathbf{e}^x - 2\mathbf{e}^y$. (1P)

(Σ = 12P)

To be handed in on Wednesday, November 20, at the beginning of the tutorial.