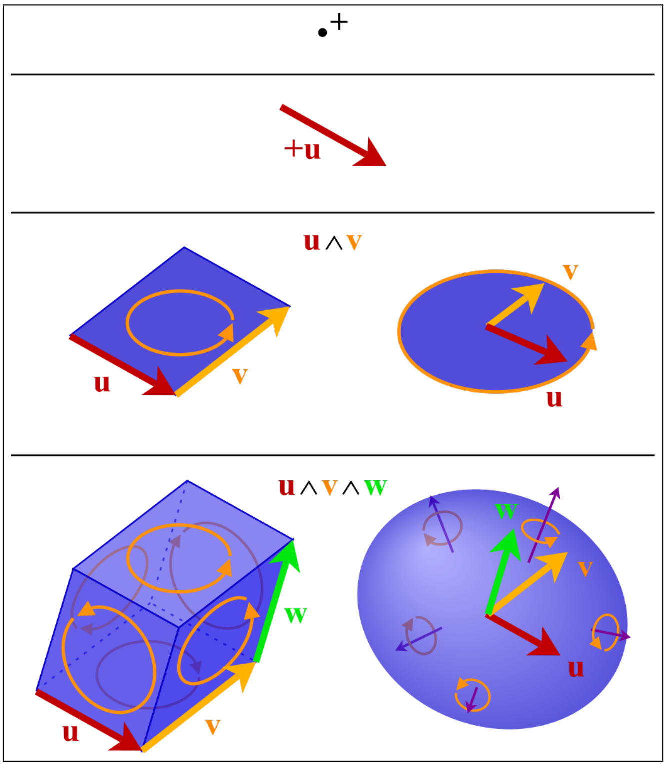

The wedge product represents oriented geometric building blocks [Wikimedia]

Exercise 4.1: Associativity of the wedge product (4P)

Let $\mathcal{A}$ be the complete antisymmetrization operator acting on a p-form ω by

$$\mathcal{A}[\omega](\mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_p) \;=\; \frac{1}{p!} \sum_{\sigma \in P_p} \mathrm{sgn}(\sigma)\, \omega(\mathbf{v}_{\sigma_1} \otimes \ldots \otimes \mathbf{v}_{\sigma_p}) \;=\; \frac{1}{p!} \sum_{i_1,\ldots,i_p} \epsilon_{i_1,\ldots,i_p} \omega(\mathbf{v}_{i_1} \otimes \ldots \otimes \mathbf{v}_{i_p})$$

The wedge product of a p_1-form ω and a p_2-form η is defined by

$$\omega \wedge \eta \;:=\; \frac{(p_1+p_2)!}{p_1!p_2!} \mathcal{A}[\omega \otimes \eta]$$

(a) Show that $\mathcal{A}$ is idempotent, i.e., $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$. (2P)

(b) Let $\alpha, \beta, \gamma \in V^*$ be 1-forms. Show that their wedge product is associative, i.e., (2P)

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).$$

Exercise 4.2: Contraction of symmetric/antisymmetric tensors (2P)

Show that the contraction of a symmetric tensor of rank (2,0) with an antisymmetric tensor of rank (0,2) always gives zero.

Please turn over ⇒

Exercise 4.3: The cross product in $\mathbb{R}^3$ **(2P)**

Let $\alpha, \beta \in \mathbb{R}^{3*}$ be two 1-forms in the Euclidean three-dimensional space. Show that the coordinates of $\alpha \wedge \beta$ in the canonical basis coincide up to a sign with the covariant components of the cross product $\alpha^\sharp \times \beta^\sharp$.

Exercise 4.4: Decomposability of 2-forms **(2P)**

An antisymmetric 2-form $\gamma \in \bigwedge^2 V^*$ is said to be *decomposable* if it factorizes into a wedge product of two 1-forms, i.e. $\gamma = \alpha \wedge \beta$.

Consider the 2-form $\gamma = \mathbf{e}^1 \wedge \mathbf{e}^2 + \mathbf{e}^3 \wedge \mathbf{e}^4$, where $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \mathbf{e}^4\}$ is the dual basis of $V^* = \mathbb{R}^4$. Is γ dedomposable or not?

Exercise 4.5: Towards the Hodge duality **(2P)**

In the Euclidean $\mathbb{R}^n$ the vector fields $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{N-1}$ define the following 1-form:

$$\varphi = \det[\,\cdot\,, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}]\,.$$

That is, applying this 1-form to the vector field $\mathbf{u}$ one obtains the result

$$\varphi(\mathbf{u}) = \det[\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}]\,,$$

where the determinant has to be formed from the components of the column vectors in an orthonormal basis. Use the volume form $\boldsymbol{\omega}$ to express the 1-form φ in an abstract way, i.e., without the use of a representation in a particular basis.

$(\Sigma =$ **12P**$)$

To be handed in on Wednesday, November 13, at the beginning of the tutorial.