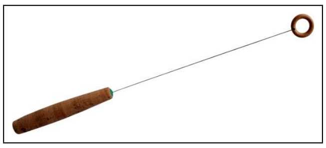
Theory of General Relativity

Lecture and Tutorials – Prof. Dr. Haye Hinrichsen / M.Sc. Alexandre Alvarez – WS 2019/20



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Exercise 3.1: Calculate components of tensor products (2P)

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors and let α and β be two 1-forms. Write down the coordinate representation of the tensors $\mathbf{u} \otimes \mathbf{v}$, $\alpha \otimes \mathbf{v}$, $\alpha \otimes \beta$, and $\alpha \otimes \mathbf{v} \otimes \beta$ in a given basis $\{\mathbf{e}_i\}$ and the corresponding co-basis $\{\mathbf{e}^j\}$.

Exercise 3.2: Preservation of symmetry/antisymmetry (2P)

Let $\mathbf{T}$ be a symmetric (or antisymmetric) 2-Form, that is $T_{ij} = \pm T_{ji}$. Show that symmetry and antisymmetry are properties that are preserved under basis transformations. In other words, prove that symmetry and antisymmetry are representation-independent features of tensors.

Exercise 3.3: Tranformation behavior of tensors (2P)

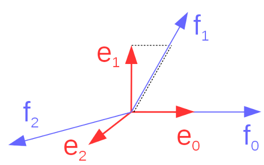
Let $\mathbf{T}$ be a 2-form and $\mathbf{v}$ a vector with the representations $\mathbf{T} = T_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$ and $\mathbf{v} = v^i \mathbf{e}_i$. Convince yourself that $\alpha = \alpha_i \mathbf{e}^i$ defined by $\alpha_i = T_{ij} v^j$ is a 1-form by showing that its components α_i transform correctly under basis transformations.

Exercise 3.4: The Gradient (2P)

Define the gradient by $\nabla := \mathbf{e}_i \partial^i$ where $\partial^i := \frac{\partial}{\partial x_i}$. Compute $g(\nabla, \mathbf{x})$ and use this result to show that ∂^i transforms like the components of a vector under basis transformations.

Exercise 3.5: The tetrad (4P)

In general relativity, the metric tensor can always be transformed in such a way that it turns locally into a Minkowski metric at a given point. Physically this would correspond to a local intertial frame (free fall) at this particular point. The purpose of this exercise is to construct the corresponding orthonormal basis, called *tetrad* (Vielbein).



Consider a 1+2-dimensional space-time equipped with the metric $\mathbf{g}$. Furthermore, let $\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2\}$ be a (non-orthonormal) basis in which $\mathbf{g}$ is represented by the matrix

$$g_{\mu\nu} = \mathbf{f}_\mu \cdot \mathbf{f}_\nu = \begin{pmatrix} -4 & 3 & -1 \\ 3 & 4 & -1/2 \\ -1 & -1/2 & 1 \end{pmatrix}, \qquad \mu, \nu = 0, 1, 2.$$

Please turn over ⇒

Go back to your linear algebra textbook and recall the Gram-Schmidt orthogonalization procedure. Use this method (starting with the time-like coordinate 0) to construct an orthonormal basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\}$, the so-called tetrad, in which the metric tensor is represented by a Minkowski metric, i.e. (3P)

$$g_{ab} = \eta_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a, b = 0, 1, 2.$$

Specify the so-called *tetrad components* ${e_a}^\mu$ of the transformation law $\mathbf{e}_a = {e_a}^\mu \mathbf{f}_\mu$. (1P)
Note: In general relativity one often uses Greek indices $\mu, \nu, \ldots$ for general coordinates and Latin indices $a, b, \ldots$ for the local Minkowski coordinates.

(Σ = 12P)

To be handed in on Wednesday, November 06, at the beginning of the tutorial.