Exercise 2.1: Conjugation in group theory (5P)

Let G be a group and let $g \in G$.

(a) Show that the conjugation $\Omega_g : h \mapsto g^{-1}hg$ is a group automorphism on G. (2P)

(b) Prove that the map $G \mapsto Aut(G) : g \mapsto (h \mapsto ghg^{-1})$ is a group homomorphism. (1P)

(c) Consider the group of permutations of three elements S_3. Show that the group $Aut(S_3)$ of conjugation automorphisms on S_3 is isomorphic to S_3, i.e., (2P)

$$Aut(S_3) \cong S_3.$$

Exercise 2.2: Function Spaces (7P)

Consider the space $\mathcal{H}$ of differentiable square-integrable $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$. With linear combinations of the form

$$h = \lambda f + \mu g \qquad \big(f, g, h \in \mathcal{H}, \quad \lambda, \mu \in \mathbb{R}\big)$$

we can regard this function space as a vector space. As physicists we extend this space by the Dirac δ-function and its derivatives. The purpose of this exercise is to show that one can deal with functions in a similar way as with finite vectors. For simplicity we shall use a very similar notation.

(a) Define the "position basis" $\{e_x\}$ with $e_x \in \mathcal{H}$, representing a function $f \in \mathcal{H}$ by

$$f = \int_{-\infty}^{+\infty} \mathrm{d}x \, f^x e_x \,,$$

where $f^x := f(x)$. What kind of function is e_x, in other words, what is $e_x(y)$? (1P)

(b) Let $\{e^y\}$ be the corresponding dual basis of the covector space $\mathcal{H}^*$ and let $f \in \mathcal{H}$. Compute $e^y(f)$. (1P)

(c) Let $\alpha \in \mathcal{H}^*$ and $f \in \mathcal{H}$. Find the representation of $\alpha(f)$ in the position basis in terms of the components α_y and f^x. (1P)

(d) Every linear map $A : \mathcal{H} \to \mathcal{H}$ can be represented by a *kernel function* $A(x,y) = A^x{}_y$. Compute the kernel functions of the linear operators $\mathbb{1}$, $\frac{\mathrm{d}}{\mathrm{d}x}$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, and $\exp(\frac{\mathrm{d}}{\mathrm{d}x})$. (2P)

(e) Consider the basis transformation $\{e_i\} \to \{\bar{e}_k\}$ defined by

$$\bar{e}_k \; = \; \int_{-\infty}^{+\infty} \mathrm{d}x \, e_x \, \tilde{M}^x{}_k \qquad \text{with} \quad \tilde{M}^x{}_k = \frac{1}{\sqrt{2\pi}} e^{-ikx}.$$

Find the corresponding transformation for the components $f^x \to \bar{f}^k$ of a function $f \in \mathcal{H}$. How are the linear operators in part (d) represented in this basis? (2P)

(Σ = 12P)

To be handed in on Wednesday, October 30, at the beginning of the tutorial.